Perception, Intuitions, and Mathematics. A Kantian Perspective

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Abstract

Traditionally, intuitions stand for images that our mind's eye represents without any help of perceptions. In mathematics, the appeal to intuitions aims to furnish a stronger foundation, for instance, of geometric constructions, whose figures can be visualized completely in imagination. In his *Aesthetic*, Kant seems to agree with this classic view. However, in his *Doctrine of Method* and *Lecture on Metaphysics*, he reduces mathematical concepts to intuition-based constructions. "There is nothing – warns Hintikka – 'intuitive' about intuitions so defined" (1992: 23). Breaking with visual images, sensible intuitions require a non-philosophical explanation. My aim is to defend this thesis. I will discuss *1)* what is an intuition for Kant, especially in relation to perceptions and mathematical concepts, and *2)* why it must be understood logically. Then, I will show *3)* how Kant builds abstract entities such as quanta and numbers on such sensible intuitions. Finally, I will compare *4)* Kant's perceptionism to Wittgenstein's conception of 'visual field' (especially in the *Tractatus*), developed in rejection of Russell's multiple-relation theory of judgments (1913).

Intuitions are usually associated with mental pictures. After all, they are images represented by means of our mind's eye without any help of sensible impressions. In mathematics, the appeal to intuitions aims to furnish a stronger foundation, for instance, of geometric constructions, whose figures can be visualized completely in imagination. Kant seems to agree with this classic view, especially in his *Aesthetic* (Kant 2003). However, Hintikka warns against this interpretation (Hintikka 1992a). The relation between Kantian intuitions and mathematics may suggest otherwise.

Kant holds that mathematics depends on sensible intuition, indeed that mathematical claims in some way refer to this intuition. He also regards such claims as synthetic rather than analytic and grounds on them the possibility of the a priori knowledge. In Kant's *Doctrine of Method* and *Lecture on Metaphysics*, mathematical concepts are exhibited *in concreto* by means of intuitions and are ultimately reduced to intuition-based constructions. "There is nothing – says Hintikka – 'intuitive' about intuitions so defined" (Hintikka 1992a: 23). Breaking with visual images, sensible intuitions require a non-philosophical explanation. They rather call for a logical-mathematical characterization.

My aim is to defend this thesis. I'll discuss 1) what is an intuition for Kant, especially in relation to mathematical concepts, and 2) why it must be understood logically. Then, I'll show 3) how Kant builds abstract entities such as quanta and numbers on sensible intuitions. Finally, I'll answer a couple of objections regarding the logical nature of intuitions and derived mathematical entities.

1. Intuitions and Mathematical Concepts

Intuition is a type of representation by means of which our mind can relate to or be conscious of objects. Since everything in our mind is a representation, Kant distinguishes intuitions from other types of representations such as perceptions and concepts:

> The genus is *representation* in general (*repraesentatio*). Subordinate to it stands representation with consciousness (*perceptio*). A *perception* which relates solely to the subject as the modification of its state is *sensation* (*sensatio*), an objective perception is *knowledge* (*cognitio*). This is either *intuition* or *concept* (*intuitus vel conceptus*). The former relates immediately to the object and is single, the latter refers to it mediately by means of a feature which several things may have in common. (Kant 2003: A320/B376-7)

As contrasted with concepts, intuitions are first characterized by immediacy. Concepts relate to objects only mediately, they contain certain properties that are possessed by those objects. In this sense, concepts represent the common features shared by several objects.

Therefore, they are called "a universal (*repraesentatio per notas communes*) or *reflected* representation (*repraesentatio discursiva*)" (*The Jäsche's Logic*, 91 §1; Kant 1992). On the opposite, intuitions are singular representations (*repraesentatio singularis*). They have only one individual object and relate to it immediately: "In whatever manner and by whatever means a mode of knowledge may relate to objects, *intuition* is that through which it is in immediate relation to them, and to which all thought as a means is directed" (A16/B33). "Thus far – says Parsons (Parsons 1992: 44) – the distinction [between intuitions and concepts] corresponds to that between singular and general terms."

Immediacy comes with singularity. The two characterize the nature of sensible intuitions only if they stay together. An intellectual intuition, for instance, would satisfy the immediacy criterion but not the singularity one. The immediacy of intuitions consists in representing their objects without relying on those properties that are shared by these objects. Concepts can be singular as well, but only as mediate representations.

In this way, concepts contrast with intuitions. However, they are also closely related. In fact, the classification of concepts depends on the distinction between empirical and pure intuitions. Let's first address this distinction.

Intuitions turn empirical as sensation comes into play. This latter is a posteriori since it derives from an affection: "that intuition which is in relation to the object through sensation, is entitled *empirical*"; whereas representations "in which there is nothing that belongs to sensation" (Kant 2003: A20/B34) are *pure*, namely pure intuitions. Therefore, the access to pure intuition requires a process of abstraction:

> If I take away from the representation of a body that which the understanding thinks in regard to it, substance, force, divisibility, etc., and likewise what belongs to sensation, impenetrability, hardness, colour,

> etc., something still remains over from this empirical intuition, namely, extension and figure. These belong to pure intuition, which, even without any actual object of the senses or of sensation, exists in the mind *a priori* as a mere form of sensibility. (Kant 2003: A20-1/B35)

After abstraction from anything empirical, something remains in every representation, namely a pure intuition. If an object is a collection of representations, each representation occupies a place in space; if abstracted from anything concrete, the collection itself (synthesis) nonetheless remains along with its ideal places. These ideal spaces are pure intuitions that, accordingly, behave like placeholders. Thus in the *Aesthetic*, an intuition is intended "as containing an infinite number of representations *within* itself", while a concept is thought of "as a representation which is contained in an infinite number of different possible representations (as their common character), and which therefore contains these *under* itself" (Kant 2003: A25/B40).

At this point, Kant distinguishes the mathematical concepts from the philosophical ones, and shows them to be an alternative way of conceptualizing: "philosophical knowledge considers the particular only in the universal, mathematical knowledge the universal in the particular, or even in the single instance, though still always *a priori* and by means of reason" (Kant 2003: A714/B742). Thus, "philosophical knowledge [...] has always to consider the universal *in abstracto* (by means of concepts), mathematics can consider the universal *in concreto* (in the single intuition) and yet at the same time through pure *a priori* representation" (Kant 2003: A734-5/B762-3).

Mathematical concepts rely on intuitions. Kant holds that "in mathematics [...] the concepts of reason must be forthwith exhibited *in concreto* in pure intuition" (Kant 2003: A711/B739), therefore "to *construct* a concept means to exhibit *a priori* the intuition which corresponds to it" (Kant 2003: A713/B741). Thus, mathematical concepts are reduced to intuition-based constructions. And these intuitions must instantiate universality under individuality (i.e., exhibit the concept *in concreto*). In this sense, pure intuitions behave logically, namely as individual representations that stand for other representations. Kant's philosophy of mathematics stands or falls on this. Let's consider a few suggestions regarding how sensible intuitions are to be properly intended.

2. Intuitions as Variables

In his *Doctrine of Method*, Kant looks to the Euclidean model. He notices that the truth value of geometric propositions runs from one claim to another "through a chain of inferences guided throughout by intuition" (Kant 2003: A716/B744). Any inference is both synthetic and evident, but none of them comes from experience since their synthesis carries on strict and not merely comparative universality. Euclidean claims are not valid generalizations derived from Hume's custom-induced inferences.

> For the construction of a concept we therefore need a *non-empirical* intuition. The latter must, as intuition, be a *single* object, and yet none the less, as the construction of a concept (a universal representation), it must in its representation express universal validity for all possible intuitions which fall under the same concept. Thus I construct a triangle by representing the object which corresponds to this concept [...]. The single figure which we draw is empirical, and yet it serves to express the concept, without impairing its universality. (Kant 2003: A713-4/B741-2)

Non-empirical intuitions clearly realize the idea that a single object or individuality may stand for a manifold of objects or universality, which is exactly the idea of free variable (x, y, z), for instance in first-order quantificational logic.

This argument is suggested by Beth, developed by Hintikka, and discussed by Parsons. In the proof that the base angles of an isosceles triangle are equal, Beth was the first to notice that:

> We proceed, as is well known, as a rule as follows first we consider a particular triangle, say ABC, and suppose that AB=AC; then we show that ∠ABC=∠ACB and have thus proved that the assertion holds in the particular case in
>
> question. Then one observes that the proof is correct for an arbitrary triangle, and therefore that the assertion must hold in general. (Beth 1957, 365)

Parsons reads Beth's argument as a case of universal generalization (UG), where we want to prove $(x)(Fx \supset Gx)$. Therefore, we assume a particular *a* such that *Fa*, deduce *Ga*, and obtain $Fa \supset Ga$ independently of the hypothesis; but since *a* was arbitrary, $(x)(Fx \supset Gx)$ follows. Hintikka rather focuses on the existential instantiation (EI): $(\exists x)Fx/Fa//p$. But both UG and EI, says Parsons, turn on "the use of a free variable which indicates *any* one of a given class of objects, so that an argument concerning it is valid for *all* objects of the class" (Parsons 1992: 55). Thus, in modern logic, pure intuitions behave like instantiations. In fact, argues Hintikka6, by instantiation methods "we introduce a representative of a particular entity a priori, without there being any such entity present or otherwise given to us" – this logical characterization of Kantian intuitions, he concludes, "has been misunderstood almost universally" (Hintikka 1992b: 345f).

The possibility of a priori knowledge relies on this use of a singular term as representative. Beth's triangle serves as a paradigm of all triangles: although it is itself an individual triangle, "nothing is used about it in the proof which is not also true of all triangles" (Parsons 1992: 61). In this case, constructing such a triangle cannot appeal to any object, it is rather a construction of concepts in pure intuition. Shabel (2006) correctly points out that constructing a single triangle provides a pattern for triangles in general, and then for all of them; since it instantiates a universal rule in a single figure, which nevertheless is ultimately made of non-empirical intuitions (intuitions without a reference to objects). Shabel's thesis is consistent with the distinction emphasized by Guyer (1987) between image and schema: "the concepts of number and triangle are [...] rules, not images of any sort" because "it is schemata, not images of objects, which underlie our pure sensible concepts" (Kant 2003: A141/B180).

> No image could ever be adequate to the concept of a triangle in general. It would never attain that universality of the concept which renders it valid of all triangles, whether right-angled, obtuse-angled, or acute-angled; it would always be limited to a part only of this sphere. The schema of the triangle can exist nowhere but in thought. It is a rule of synthesis of the imagination, in respect to pure figures in space. (Kant 2003: A141/B180)

Thus, Kant's mathematical method turns on constructions. It consists, says Hintikka, in "introducing particular representatives of general concepts and carrying out arguments in terms of such particular representatives, arguments which cannot be carried out by means of general concepts" (Hintikka 1992b: 24). In fact, argues Parsons, the algebraist's "manipulating symbols according to certain rules [requires] analogous intuitive representation of his concept", and that "the symbolic construction is essentially a construction with *symbols* as objects of intuition" (Parsons 1992: 65). The same conclusion is independently reached by Friedman: "from a modern point of view, we could perhaps reconstruct Kant's conception of arithmetic as involving a sub-system of primitive recursive arithmetic (such as Robinson arithmetic) where generality is expressed by means of free variables and there are no true quantifiers" (Friedman 1992, 113).

In what follows, I'll briefly show how Kant builds quanta and numbers on the notion of sensible intuitions.

3. Quanta and Numbers

As we just saw, Kant conceives mathematics and its concepts in terms of intuition-based constructions. Then, he combines these intuitions and derives *quanta,* namely quantified parts or properties. Recall that in modern logic, binding a variable that ranges over a domain is called 'quantification'. Therefore, if intuitions stand for free variables, they are supposed to be quantified. Let's take a closer look.

According to the Kantian variation of Hume's bundle theory, an object is a collection of representations (properties), each of them corresponding to an intuition. The same collection (concept) can be seen either as empirical or as pure, depending on (the presence or absence of) sensations. But if it is the case, what kind of object does derive from the synthesis of such a pure manifold? What object is made by pure intuitions alone? Kant answers straightforwardly:

> As regards the formal element, we can determine our concepts in *a priori* intuition, inasmuch as we create for ourselves, in space and time, through a homogeneous synthesis, the objects themselves – these objects being viewed simply as *quanta*. (Kant 2003: A723/B751)

From the synthesis of the pure manifold derive *quanta*, namely objects (wholes) whose parts allow for quantification. Each of these parts corresponds to a pure intuition, which is, therefore, thought of as quantifiable. Kant's leading idea is that combining (*zusammensetzen*) homogeneous parts leads to magnitudes (Kant 2001; 29: 991, 1794-5) – an idea borrowed from Euclid (see Sutherland 2004). Such combining is to be understood in terms of proportions or ratios, though. In order to be either bigger or smaller or equal, two parts must be conceived as inside of one another, that is, as part and whole: "A > than B if a part of A=B; in contrast A < B, if A is equal to a part of B" (28: 506, late 1780s) or "something is larger than the other if the latter is only equal to a part of the former" (28: 561, 1790-1). This explains quantity in terms of *part-whole relations* and *homogeneity*.

At this point, Kant can directly develop the notion of quanta into that of number. He just needs to differentiate between ostensive (geometric) and symbolic (arithmetic) constructions.

> But mathematics does not only construct magnitudes (*quanta*) as in geometry; it also constructs magnitude as such (*quantitas*), as in algebra. In this it abstracts completely from the properties of the object that is to be thought in terms of such a concept of magnitude. (Kant 2003: A717/B745)

Numbers[1] are homogeneous parts (quanta) combined in succession. In his *Metaphysics* L_2 (28: 561, 1790-1), Kant argues that "each quantum is a multitude [and] must thus also consist of homogeneous parts" and that, as such, "each quantum can be increased or decreased". This goes through combining its parts, "the parts that, connected (*verbunden*) with each other, make a number concept". In this mereological connection "something is larger than the other if the latter is only equal to a part of the former"; in fact, "for something to alter into a larger is to increase, and for something to alter into a smaller is to decrease".

Recall that "the *subsumption* of intuitions under pure concepts" (Kant 2003: A138/B177) follows certain rules, which are called by Kant schemata. Therefore, "the pure *schema* of magnitude (*quantitatis*), as a concept of the understanding, is *number*, a representation which comprises the successive addition of homogeneous units" (A142/B182).

If sensible intuitions stand for logical placeholders, Kant's notion of number could be accommodated in the set-theoretic way described by Benacerraf's Ernie theorem (Benacerraf 1965, 54), "for any two numbers, *x* and *y*, *x* is less than *y* if and only if *x* belongs to *y* and *x* is a proper subset of *y*" rather than Johnny's "given two numbers, *x* and *y*, *x* belongs to *y* if and only if *y* is the successor of *x*" – i.e., $0=\emptyset$, $1=\{0\}=\{\emptyset\}$, $2=\{0, 1\}=\{\emptyset,\{\emptyset\}\}$, $3 =\{0, 1, 2\}=\{\emptyset,\{\emptyset\},\{\emptyset,\{\emptyset\}\}\}$, and so on. If this is right, the mathematical intuitionism of Kant somehow anticipates Cantor's theory of sets and opposes Dedekind's relationism (including his fellow neo-Kantians).

4. Conclusions

In conclusion, let's address a couple of objections. As most of us hold, there are individual variables (intuitions) of algebra that do not entail any relation to our sensibility. We can know individuals and do not involve sense-perceptions, e.g., in dealing with numbers and other abstract entities. In this case, "*a priori* intuitions – says Hintikka – are not characterized by an especially immediate relation to their objects; they are precisely intuitions used in the absence of their objects" (Hintikka 1992b: 358) and can hardly be intended as perceptual. However, this possibility is explicitly ruled out by Kant in the *Aesthetic*, where all intuitions (including those used in mathematics) are seen as *sinnlich* and reduced to *passive* perceptions. But, if Kantian intuitions really stand for variables, why are they sensible?

Hintikka dismisses the *Aesthetic*. In his eyes, Kant's philosophy of mathematics exclusively derives from his *Doctrine of Method*, where sensible intuitions stand for logical instantiations. Arbitrarily chosen representatives of general concepts are introduced a priori into mathematical claims, as Kant says: "our new method of thought, namely, that we can know a priori of things only what we ourselves put into them" (Kant 2003: Bxviii). By means of this instantiation method (identified by Hintikka with EI: $(\exists x)Fx/Fa//p$), we anticipate certain properties and relations of particulars. Then, "*we have ourselves put those properties and relations into* objects in the processes through which we come to know individuals (particulars)" (347). These processes are carried out by our sense-perceptions. Hence, those properties and relations are due to the structure of our sensibility, namely space and time. Kant draws a legitimate conclusion.

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[1] Only *quanta* whose magnitude is *extensive* qualify as numbers (*discrete quanta*). The parts of *continuous quanta* are indeterminate and their magnitude is *intensive*, namely given by a degree.

Parsons endorses this solution. After all, the symbols implied by conceptual constructions in intuition are perceptible objects. Those constructions need something phenomenological like perceptions (represented by single instances).

A correlated issue concerns the nature of mathematical objects. Kant does not explicitly grant existence to them. He rather takes 'existence' as a concrete attribute, ultimately perceivable. "But what – asks Parsons – are a priori intuitions, as singular representations, intuitions *of*?" (Parsons 1992: 73). In other terms, if mathematics contains a priori knowledge (which is knowledge of objects), what kind of objects does it really know? A suggestion may be to postulate *abstract* entities "beyond the field of possible experience" and to construct them as "in arithmetic and predicative set theory", namely "as *forms* of spatiotemporal objects" (Parsons 1992: 64).

After all, the object dependence doesn't hold for intuitions whose nature isn't empirical but logical-mathematical.

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